Average Incremental Cost Pricing for the AC Unit Commitment Problem

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Abstract

Unit Commitment (UC) problems that consider the Alternating Current (AC) model of the transmission network have long been considered intractable to solve at scale by the power system community. Recently, the Grid-Optimization (GO) Competition held by the Advanced Research Project Agency-Energy (ARPA-E) has facilitated the development of the first algorithms to solve large-scale ACUC problems. This new capability opens a path towards the explicit consideration of the AC transmission network model in UC problems used to clear day-ahead electricity markets. This calls for the analysis of electricity market structures that accommodate both the continuous non-linearity of the AC transmission network and the discrete non-linearity of the UC problem simultaneously. This paper serves as an initial effort to do so by proposing an Average Incremental Cost (AIC) pricing structure that is designed around the ACUC problem. In particular, an AIC one-pass pricing problem is proposed that represents a continuously constrained variant of the ACUC problem and allows for the computation of Locational Incremental Prices (LIPs) for both real and reactive power as the local optimal Lagrange multipliers of the power balance constraints. To avoid degeneracy, the pricing problem includes a small parameter $\epsilon > 0$. Under certain assumptions market participants are shown to realize profit that converges to a non-negative value as ϵ approaches zero, practically ensuring profitability for small values of ϵ. We additionally provide many simple and important examples that provide intuition and insights into the proposed prices. Examples illustrate the basic concept of AIC pricing, the derived profitability results, the existence of multiple LIPs, the importance of including reactive power in the dispatch and pricing problems, the need for reactive power prices, and the improved incentives exhibited by LIPs as compared to traditional Locational Marginal Prices (LMPs). We additionally indicate many directions for future work including analysis of larger test cases.

1 Introduction

Wholesale electricity markets serve over two-thirds of the consumers in the United States. These markets are operated by Independent System Operators (ISOs) that dispatch generation and determine electricity prices. The generation dispatch is determined by solving a large scale economic dispatch optimization problem and prices are traditionally chosen to represent the marginal cost of serving load. For the special case where the economic dispatch problem is convex, these Locational Marginal Prices (LMPs) efficiently clear and settle the market; however, practical economic dispatch problems are non-convex, due to the underlying non-linear physics of the electric power system. As a result, LMPs often cause market inefficiencies including but not limited to the need for non-transparent side-payments, also known as make-whole payments, to market participants by the

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ISO to support dispatch decisions and revenue insufficiency of market participants and the ISO. Seminal work acknowledges that no nondiscriminatory pricing structure exists that addresses all problems associated with the non-convexity observed in electricity markets and instead propose pricing structures that aim to minimize the inefficiencies [1, 2, 3]. ISO interest in this topic along with recent advancements in computational capabilities have fueled a resurgence in non-convex pricing research that proposes different variations of electricity prices and algorithms to compute those prices [4, 5, 6, 7, 8, 9].

The two most commonly studied sources of non-convexity associated with power system economic dispatch problems, extending beyond applications of electricity markets, are those that arise from binary unit-commitment variables representing generator on/off statuses and those that arise from the non-linear model of the Alternating Current (AC) transmission network. These two sources of non-convexity are typically studied separately in the form of the Unit Commitment (UC) problem and the AC Optimal Power Flow (AC OPF) problem respectively [10, 11]. These two problems are individually difficult to solve in general from a computational perspective. More specifically, general instances of these problems have been proven to be NP-hard [12, 13]; however, practical instances of these problems tend to be tractable to solve to a reasonable optimality tolerance. Furthermore, combining the two problems into one AC UC problem has previously been considered intractable to solve by the power system community. For this reason, previous works regarding non-convex pricing structures have only studied pricing structures in the context of either the UC problem or the AC OPF problem with much more focus on the UC problem, e.g. [7, 14, 15, 16]. The Grid-Optimization (GO) Competition held by the Advanced Research Project Agency-Energy (ARPA-E) [17] recently facilitated the development of the first algorithms to solve large scale economic dispatch problems containing both binary unit-commitment variables and the AC transmission network model, a problem that we will refer to as the ACUC problem. This new capability calls for the analysis of pricing structures that accommodate both aforementioned sources of non-convexity simultaneously. This paper serves as an initial effort to do so.

This paper focuses on extensions to the recently proposed Average Incremental Cost (AIC) pricing structure that accounts for incremental costs associated with generator commitments and guarantees profitability of dispatched market participants [15]. The AIC pricing structure is similar to the marginal pricing structure and produces Locational Incremental Prices (LIPs) that are analogous to LMPs. Intuitively, LIPs represent the incremental cost of serving load. In a simple copper-plate test case with one time interval and fixed demand the LIP represents the total cost of the most costly generator averaged over its real power output, see the example in Section 4.1.1. AIC pricing has only been studied in the context of the UC problem to date while previous work has neglected any non-linearity associated with the AC transmission system. We particularly extend the AIC one-pass pricing methodology that generalizes the concept of AIC pricing to a setting with network constraints and multiple time intervals while also facilitating efficient computation by solving a pricing problem that represents a continuously constrained variant of the ACUC problem and yields optimal prices as local optimal Lagrange multipliers [18]. We extend previous work by considering the AC transmission constraints in the UC problem and by pricing reactive power as a market product.

Convex hull pricing is another pricing structure that deserves attention in future work. Convex Hull Prices (CHPs) are locational prices chosen to minimize *uplift* quantities that represent inefficiency metrics such as side-payments and potential congestion revenue shortfall. CHPs are not easy to compute in general and different methods of computing these prices have been proposed. In the context of the AC OPF problem and UC problem (separately) CHPs have been approximated by formulating and solving a primal pricing problem that uses convex approximations of the convex hull of certain sets [14, 7]. In the context of the UC problem Dantzig-Wolf decomposition and branch and bound methods have shown promise in computing exact CHPs; however, it is not clear that these methods will extend to the ACUC problem [9, 8]. We choose to focus on AIC pricing, in part, because LIPs are relaitvely easy to compute through the AIC one-pass pricing problem. Additionally, LIPs improve upon CHPs in a number of ways outlined in [15] including but not limited to the elimination of side-payments, the improvement in market entry and exit signals, and the consistency in the application with ISO penalties imposed on market-participant deviation from dispatch instructions. Additionally, LIPs do not exhibit common criticisms of CHPs such as the ability of market-participants that are offline to set prices, the assumption that make-whole payments may compensate offline generators to not self-dispatch, and the use of definitions of various uplift quantities that often do not represent uplift definitions used in practice.

This paper is organized as follows. Section 2 presents a general form of the ACUC problem as a Mixed Integer Non-Linear Program (MINLP) that generalizes the transmission network and market participant models. Section 3 presents the AIC one-pass pricing problem which represents a continuously constrained variant of the ACUC problem and yields LIPs as the local optimal Lagrange multipliers of the power balance constraints. To avoid a common degeneracy that causes the existence of multiple LIPs, an ϵ parameter is introduced that effectively relaxes the AIC one-pass pricing problem. Section 3 continues by informally showing that market participants will realize non-negative profits under certain assumptions that often hold. This profitability result is presented in terms of a limit as the ϵ parameter approaches zero. Section 4 presents simple examples that provide intuition behind AIC pricing and illustrate what happens when some of the assumptions are violated. Simple examples also justify the importance of including AC transmission constraints in both the ACUC and AIC one-pass pricing problems and justify the inclusion of the ϵ parameter to encourage uniqueness of LIPs. This paper does not address computation effort and scalability of solving the AIC one-pass pricing problem and does not compare LIPs to CHPs, which is left for future work.

2 Alternating Current Unit Commitment (ACUC) Problem

This section provides a general formulation of the ACUC problem. We begin with notation. Throughout this paper scalars and scalar functions are denoted with lower case letters, vectors and vector valued functions are denoted with bold lower case letters, and matrices are denoted with capital letters. Vectors are represented as column vectors. The column vector of ones and zeros are denoted $\mathbf{1}$ and $\mathbf{0}$ and are of appropriate dimension. The set of integers from a to b is denoted $[a, b]$. Row k of matrix G is denoted G_k and represents a row vector. Column i of matrix G is denoted $G_{[i]}$ and represents the column vector. The element of matrix G in the k^{th} row and i^{th} column is denoted $G_{k,i}$. The transpose of matrix G is denoted $G^\dagger$.

The transmission network has n buses indexed by $[1, n]$ and the ACUC problem has τ time intervals indexed by $[1, \tau]$. The total number of generators is γ and each generator is indexed $j \in [1, \gamma]$. The real and reactive generation matrices $G \in \mathbb{R}^{\tau \times \gamma}$ and $R \in \mathbb{R}^{\tau \times \gamma}$ represent the generation for each generator at each time interval. The generator statuses are represented by $X \in \mathbb{R}^{\tau \times \gamma}$. Generator j has private constraints represented by the set $\mathcal{X}_j$ and has a cost function denoted $c_j(G_{[j]}, X_{[j]})$ that is assumed to be convex, increasing, and continuous in all arguements. The fixed real and reactive demand at each node for each time interval is denoted $D \in \mathbb{R}^{\tau \times n}$ and $L \in \mathbb{R}^{\tau \times n}$. The real and reactive power extraction from transmission lines at each node for each time interval are $P \in \mathbb{R}^{\tau \times n}$ and $Q \in \mathbb{R}^{\tau \times n}$. The transmission constraints are represented by the set $\mathcal{T}_t$ for each time interval t.

The following Alternating Current Unit Commitment formulation is referenced as ACUC.

$$\min_{\substack{(G,R)\in\{\mathbb{R}^{\tau\times\gamma}\}^2,\ X\in\{0,1\}^{\tau\times\gamma} \\ (P,Q)\in\{\mathbb{R}^{\tau\times n}\}^2}} \sum_{j=1}^{\gamma} c_j(G_{[j]}, X_{[j]}) \tag{2.1}$$

$$G_t M - D_t - P_t = \mathbf{0} \ \forall t \in [1, \tau] \tag{2.1a}$$

$$R_t M - L_t - Q_t = \mathbf{0} \ \forall t \in [1, \tau] \tag{2.1b}$$

$$(G_{[j]}, R_{[j]}, X_{[j]}) \in \mathcal{X}_j \ \forall j \in [1, \gamma] \tag{2.1c}$$

$$(P_t, Q_t) \in \mathcal{T}_t \ \forall t \in [1, \tau] \tag{2.1d}$$

ACUC minimizes total costs subject to various constraints. Constraints (2.1a) and (2.1b) represent the real and reactive power balance constraints at each bus for each time interval. The matrix $M \in \mathbb{R}^{\gamma \times n}$ maps generators to busses such that $M_{j,i} = 1$ if generator j is located at bus i and $M_{j,i} = 0$ otherwise. Note that the total generation at bus i during time interval t is $G_t M_{[i]}$. Constraints (2.1c) represent private constraints for each of the market participants. Constraints (2.1d) represent transmission constraints for each of the time intervals.

ACUC is a Mixed Integer Non-Linear Program (MINLP) where the integrality constraints for commitment statuses are enforced implicitly as the constraint $X \in \{0, 1\}^{\tau \times \gamma}$. The private constraints for generator j, denoted $\mathcal{X}_j$, is enforced separately and may be convex. Indeed, this set is often a polytope in practice. Importantly, our generator model represents a more general market participant capable of generating or consuming and can accommodate storage devices or flexible demand. Indeed, the example in Section 4.1.3 uses this general model to represent flexible demand. Nevertheless, throughout the paper we often refer to this general market participant as a generator.

This problem formulation is a generalization of the traditional UC problem that uses the linear Direct Current (DC) approximation for the transmission constraints. The *DCUC problem* is a special case of the ACUC problem where $\mathcal{T}_t$ represents a linearly constrained set and reactive power Q_t is unconstrained. As a result, the constraint (2.1b) is always satisfied and thus we can neglect reactive power by removing (2.1b) along with the reactive power arguments in constraints (2.1c) and (2.1d). Furthermore, there are no real power losses for the DC transmission network, and thus all feasible real power injections P_t satisfy $P_t \mathbf{1} = 0$. The DCUC problem is a Mixed Integer Linear Program (MILP).

It is very difficult to solve large instances of the ACUC problem (2.1). In fact, this problem is a generalization of a simpler UC problem that is itself strongly NP-hard [12]. As a result, we cannot expect to optimally solve the ACUC problem. Instead, we assume that a feasible point of the ACUC problem can be identified. Indeed, determining feasibility of the ACUC problem (2.1) is also a very difficult problem and is a generalization of the AC feasibility problem, which is itself strongly NP-hard [13]. Nevertheless, practical instances of this problem typically have a feasible solution that can be identified. With this in mind, we define the generation and commitment dispatch to be feasible with respect to the ACUC problem (2.1) and denote it $(G^\star, R^\star, X^\star)$.

Definition 1. The *generation and commitment dispatch* is denoted $(G^\star, R^\star, X^\star)$ and satisfies the constraints (2.1a)-(2.1d) for some values of (P, Q).

3 Average Incremental Cost (AIC) One-Pass Pricing Problem

To determine the optimal commitment and dispatch, the ISO solves the ACUC problem (2.1). To determine the prices, ISOs will then typically solve a continuously constrained variant of the ACUC problem that yields prices as the Lagrange multipliers of the power balance constraints. In the context of traditional LMPs, the *marginal pricing problem* represents (2.1) with the integer variables X fixed to the commitment dispatch $X^\star$, resulting in a continuously constrained problem with well-defined Lagrange multipliers. However, by fixing the integers the marginal pricing problem does not allow the LMPs to capture any costs related to commitment statuses, e.g. start-up costs. This motivates the introduction of AIC pricing. Similar to LMPs, LIPs are determined by solving a continuously constrained variant of the ACUC problem termed the *AIC one-pass pricing problem* that yields prices as the Lagrange multipliers of the power balance constraints. Section 3.1 formulates the AIC one-pass pricing problem and defines the LIPs as the Lagrange multipliers of the power balance constraints. Section 3.2 then provides results regarding the profitability of generators that are proven informally.

3.1 AIC One-Pass Pricing Problem

In contrast to the marginal pricing problem, the AIC one-pass pricing problem represents the ACUC problem with the integer variables X relaxed to lie in the hyper-rectangle $[\mathbf{0}, X^\star]$, making the problem continuously constrained. This problem also introduces bounds on the real and reactive power that are proportional to the commitment variables and intend to average the fixed costs over the real and reactive power. The *AIC one-pass pricing problem* is as follows:

$$\min_{\substack{(G,R)\in\{\mathbb{R}^{\tau\times\gamma}\}^2, X\in[\mathbf{0},X^\star]\\ (P,Q)\in\{\mathbb{R}^{\tau\times n}\}^2}} \sum_{j=1}^{\gamma} c_j(G_{[j]}, X_{[j]}) \tag{3.2}$$

$$(\Pi_t) \qquad G_t M - D_t - P_t = \mathbf{0} \ \forall t \in [1, \tau] \tag{3.2a}$$

$$(\Lambda_t) \qquad R_t M - L_t - Q_t = \mathbf{0} \ \forall t \in [1, \tau] \tag{3.2b}$$

$$(G_{[j]}, R_{[j]}, X_{[j]}) \in \mathcal{X}_j \ \forall j \in [1, \gamma] \tag{3.2c}$$

$$(P_t, Q_t) \in \mathcal{T}_t \ \forall t \in [1, \tau] \tag{3.2d}$$

$$\left(\text{diag}(G^\star_{[j]}) - \epsilon\right) X_{[j]} \le G_{[j]} \le \left(\text{diag}(G^\star_{[j]}) + \epsilon\right) X_{[j]} \ \forall j \in [1, \gamma] \tag{3.2e}$$

$$\left(\text{diag}(R^\star_{[j]}) - \epsilon\right) X_{[j]} \le R_{[j]} \le \left(\text{diag}(R^\star_{[j]}) + \epsilon\right) X_{[j]} \ \forall j \in [1, \gamma] \tag{3.2f}$$

The AIC pricing problem relaxes the integers X to lie in the hyper-rectangle $[\mathbf{0}, X^\star]$, making the problem continuously constrained. Constraints (3.2e) and (3.2f) represent bounds on the real and reactive power that are proportional to the commitment variables and intend to average the fixed costs over the real and reactive power market products that are being priced. To avoid degeneracy a small tolerance of $\epsilon > 0$ is introduced to these constraints. Indeed, increasing ϵ effectively relaxes the problem. Lagrange multipliers are provided in parenthesis to the left of the power balance constraints. The Lagrange multipliers are represented as matrices and are denoted $\Pi \in \mathbb{R}^{\tau \times n}$ and $\Lambda \in \mathbb{R}^{\tau \times n}$. Note that the real power price seen by generator j at time interval t is $M_j \Pi_t^\dagger$.

The AIC one-pass pricing problem (3.2) is difficult to solve in general as it enforces AC transmission constraints and, as mentioned in the previous section, the AC feasibility problem is NP-hard in general. However, for small values of ϵ, the problem may become easier due to the significantly reduced size of the feasible set. This paper does not provide analysis of the computational complexity or the computational burden of solving (3.2). However, this problem is similar to a multi-interval AC OPF problem and can likely be solved to at least a local minimum for practical cases as is typically true for the AC OPF problem.

As already mentioned, a small $\epsilon > 0$ value is introduced to (3.2) to avoid degeneracy. Indeed, if $\epsilon = 0$, then multiple LIPs often exist as shown in the example in Section 4.3.1. Section 3.2 of this paper shows that generators are profitable as $\epsilon > 0$ approaches zero to avoid the degeneracy problem, although generator profitability still holds when $\epsilon = 0$. The examples in Section 4 use very small values of ϵ; however, future work should investigate how to best choose ϵ given physical characteristics of the power system and given tolerance levels of optimization solvers used.

We will make the assumption that Karush–Kuhn–Tucker (KKT) conditions of the AIC one-pass pricing problem (3.2) can be solved and we can obtain local optimal Lagrange multipliers for the real and reactive power balance constraints denoted Π and Λ respectively, see Remark 3.1. The LIPs are then defined by a KKT point of the AIC one-pass pricing problem (3.2). Below is a general definition of a KKT point for (3.2) that follows from the normal cone definition of the First Order Necessary Conditions (FONCs), see Remark 3.2, and assumes that the generator cost functions c_j are continuously differentiable in all arguments.

Definition 2. Consider a generation and commitment dispatch $(G^\star, R^\star, X^\star)$ as in Definition 1 along with a positive scalar ϵ. An *AIC KKT point* is denoted

$$(\hat{\Pi}, \hat{\Lambda}, \hat{P}, \hat{Q}, \hat{G}, \hat{R}, \hat{X}) \in \left\{\mathbb{R}^{\tau \times n}\right\}^4 \times \left\{\mathbb{R}^{\tau \times \gamma}\right\}^2 \times [\mathbf{0}, X^\star]$$

and is such that constraints (3.2a)-(3.2f) hold along with the following conditions:

$$(-\hat{\Pi}_t, -\hat{\Lambda}_t) \in \mathcal{N}_{\mathcal{T}_t}(\hat{P}_t, \hat{Q}_t) \quad \forall t \in [0, \tau] \tag{3.3a}$$

$$-\nabla f_j(G_{[j]}, R_{[j]}, X_{[j]}; \hat{\Pi}, \hat{\Lambda})\Big|_{(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]})} \in \mathcal{N}_{\tilde{\mathcal{X}}_j}(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]}) \forall j \in [1, \gamma] \tag{3.3b}$$

where $\tilde{\mathcal{X}_j} := \mathcal{X}_j \cap \mathcal{B}_j(\epsilon) \cap \{\mathbb{R}^\tau \times \mathbb{R}^\tau \times [\mathbf{0}, X^\star_{[j]}]\}$ and $\mathcal{B}_j(\epsilon)$ is a set that depends on ϵ and is defined by (3.2e) and (3.2f) as follows:

$$\mathcal{B}_j(\epsilon) := \{(G_{[j]}, R_{[j]}, X_{[j]}) \in \{\mathbb{R}^\tau\}^3 : \text{ constraints (3.2e) and (3.2f) hold for generator } j\}$$

The function $f_j(G_{[j]}, R_{[j]}, X_{[j]}; \hat{\Pi}, \hat{\Lambda})$ is expressed as follows:

$$f_j(G_{[j]}, R_{[j]}, X_{[j]}; \hat{\Pi}, \hat{\Lambda}) := c_j(G_{[j]}, X_{[j]}) - M_j \hat{\Pi}^\dagger G_{[j]} - M_j \hat{\Lambda}^\dagger R_{[j]} \tag{3.4}$$

and its gradient evaluated at the point $(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]})$ is denoted $\nabla f_j(G_{[j]}, R_{[j]}, X_{[j]}; \hat{\Pi}, \hat{\Lambda})|_{(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]})} \in \{\mathbb{R}^\tau\}^3$. The normal cone of a general set $\mathcal{S}$ evaluated at $\hat{s}$ is denoted $\mathcal{N}_{\mathcal{S}}(\hat{s})$.

Intuitively, conditions (3.3a) and (3.3b) represent generalized stationarity conditions for problem (3.2). The LIPs for real power and the LIP for reactive power are represented by the Lagrange multipliers $\hat{\Pi}$ and $\hat{\Lambda}$ respectively from Definition 2. Indeed, it is possible that there exists multiple LIPs that satisfy the KKT conditions for a given dispatch $(G^\star, R^\star, X^\star)$. Our theoretical results do not require these LIPs to be unique; however, future work should investigate uniqueness of LIPs and the impact that non-uniqueness may have on pricing incentives in the electricity market, see the example in Section 4.3.2.

Remark 3.1. A KKT point may not exist in our general framework because constraint qualifications may not be satisfied within the generalized constraint sets; however, practical cases should satisfy some constraint qualifications that ensure existence of a KKT point. Although standard off-the-shelf software is typically able to solve the KKT conditions and identify an associated KKT point, computational issues may arise particularly for large instances of the problem that include the AC transmission network model. Furthermore, it is possible that a solution satisfying the KKT conditions could represent a saddle point, local maximum, or local minimum. However, in practice, standard off-the-shelf software typically converges to a local minimum if not a global minimum. Furthermore, the theoretical results in this paper still hold if the KKT point represents a saddle point, local maximum, or local minimum.

Remark 3.2. The normal cone definition of the FONCs are a generalization of and are typically used to derive the common KKT conditions. Appendix A in reference [19] provides a description of how the normal cone definition of the FONCs are related to the common KKT conditions. It can be easily shown that the common KKT conditions of (3.2) imply the generalized KKT conditions in Definition 2, as is shown for a different pricing problem in reference [19]; however, we do not prove this explicitly here.

3.2 Profitability of Market Participants

In practice ISOs often provide make-whole payments to market participants that realize a deficit to encourage market participation by eliminating risk of loss. However, these make-whole payments cause market inefficiency for a number of reasons. For example, make-whole payments distort price signals for generation investment because they represent private transactions not disclosed to the public and thus make it difficult for potential market entrants to predict profitability. Another problem caused by make-whole payments is that they are not covered by another revenue stream of the ISO and thus cause revenue insufficiency of the ISO.

One of the proposed key features of AIC pricing is the guarantee that market participants realize non-negative profits after market clearing and thus require zero make-whole payments. As illustrated in the example in Section 4.1.2, this result is not generally true; however, we show that it is true under certain assumptions that often hold. Assumption 1 intuitively states that generator j is capable of not participating in the market by being uncommitted and producing zero generation at zero cost. As in the example in Section 4.1.2, violations of Assumption 1 are often caused by problems associated with the rolling horizon implementation of the AIC one-pass pricing problem that can often be corrected by simply adjusting the time horizon to allow Assumption 1 to hold. Future work further analyze effective rolling horizon implementations of the AIC one-pass pricing problem.

Assumption 1. *Generator j is able to produce zero generation at zero cost at each time step.*

$$(\mathbf{0},\mathbf{0},\mathbf{0}) \in \mathcal{X}_j \text{ and } c_j(\mathbf{0},\mathbf{0}) = 0$$

The following proposition represents an intermediate result stating that each generator satisfying Assumption 1 will also satisfy (3.5). Intuitively, the condition (3.5) would ensure that generator j realizes non-negative profit if the generation and commitment from the KKT point $(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]})$ represented the dispatch to market participants. This proposition is similar to common results regarding non-negative profits for market participants in convex markets, where the AIC one-pass pricing problem (3.2) represents both the dispatch problem and the pricing problem.

Proposition 1. *Consider an AIC KKT point from Definition 2. If generator j satisfies Assumption 1, then the AIC KKT point satisfies the following inequality:*

$$M_j \hat{\Lambda}^\dagger \hat{R}_{[j]} + M_j \hat{\Pi}^\dagger \hat{G}_{[j]} - c_j(\hat{G}_{[j]}, \hat{X}_{[j]}) \geq 0 \tag{3.5}$$

Sketch of Proof: The condition (3.3b) represents the normal cone definition of the FONCs for the generator profit maximization problem where the generator private constraints are represented by $\tilde{\mathcal{X}}_j$. These FONCs can alternatively be written in terms of the tangent cone definition of the FONCs as follows:

$$\begin{aligned}
&\nabla_3 f_j(G_{[j]}, R_{[j]}, X_{[j]}; \hat{\Pi}, \hat{\Lambda})\Big|^\dagger_{(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]})} (X_{[j]} - \hat{X}_{[j]}) \\
&\quad + \nabla_2 f_j(G_{[j]}, R_{[j]}, X_{[j]}; \hat{\Pi}, \hat{\Lambda})\Big|^\dagger_{(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]})} (R_{[j]} - \hat{R}_{[j]}) \\
&\quad + \nabla_1 f_j(G_{[j]}, R_{[j]}, X_{[j]}; \hat{\Pi}, \hat{\Lambda})\Big|^\dagger_{(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]})} (G_{[j]} - \hat{G}_{[j]}) \geq 0 \quad \forall (G_{[j]}, R_{[j]}, X_{[j]}) \in \mathcal{C}_{\tilde{\mathcal{X}}_j}(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]})
\end{aligned} \tag{3.6}$$

where $\mathcal{C}_{\tilde{\mathcal{X}}_j}(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]})$ represents the tangent cone of the set $\tilde{\mathcal{X}}_j$ evaluated at the point $(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]})$ and represents all feasible directions from that point. The gradient of f with respect to its i^{th} argument is denoted $\nabla_i f$. This condition intuitively states that all feasible directions from the KKT point have a positive dot product with the gradient of the cost function and thus the KKT point represents a local optimum. Substituting explicit expressions for the gradients leads to the following:

$$\begin{aligned}
&\nabla_2\, c_j(G_{[j]}, X_{[j]})\big|^\dagger_{(\hat{G}_{[j]}, \hat{X}_{[j]})} (X_{[j]} - \hat{X}_{[j]}) - M_j \hat{\Lambda}^\dagger (R_{[j]} - \hat{R}_{[j]}) \\
&\qquad + \left(\nabla_1\, c_j(G_{[j]}, X_{[j]})\big|^\dagger_{(\hat{G}_{[j]}, \hat{X}_{[j]})} - M_j \hat{\Pi}^\dagger\right) (G_{[j]} - \hat{G}_{[j]}) \geq 0 \\
&\qquad\qquad \forall (G_{[j]}, R_{[j]}, X_{[j]}) \in \mathcal{C}_{\tilde{\mathcal{X}}_j}(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]})
\end{aligned} \tag{3.7}$$

We claim that $(\mathbf{0},\mathbf{0},\mathbf{0}) \in \mathcal{C}_{\tilde{\mathcal{X}}_j}(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]})$. This is easily proven by construction. By Assumption 1 we have that $(\mathbf{0},\mathbf{0},\mathbf{0}) \in \mathcal{X}_j$ and so $(\mathbf{0},\mathbf{0},\mathbf{0}) \in \tilde{\mathcal{X}}_j$ because $\tilde{\mathcal{X}}_j$ represents the intersection of three sets that contain the origin. Furthermore, the set $\tilde{\mathcal{X}}_j$ is convex because it is the intersection of three convex sets. As a result, the line segment connecting the point $(\mathbf{0},\mathbf{0},\mathbf{0}) \in \tilde{\mathcal{X}}_j$ to the point $(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]}) \in \tilde{\mathcal{X}}_j$ is completely contained within the set $\tilde{\mathcal{X}}_j$. This implies there is a feasible direction pointing towards the origin and thus

$$(\mathbf{0},\mathbf{0},\mathbf{0}) \in \mathcal{C}_{\tilde{\mathcal{X}}_j}(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]}). \tag{3.8}$$

By (3.8) we can now evaluate the inequality (3.7) at the point $(G_{[j]}, R_{[j]}, X_{[j]}) = (\mathbf{0},\mathbf{0},\mathbf{0})$. Doing this and rearranging terms leads to the following:

$$M_j \hat{\Lambda}^\dagger \hat{R}_{[j]} + M_j \hat{\Pi}^\dagger \hat{G}_{[j]} - \nabla_2\, c_j(G_{[j]}, X_{[j]})\big|^\dagger_{(\hat{G}_{[j]}, \hat{X}_{[j]})} \hat{X}_{[j]} - \nabla_1\, c_j(G_{[j]}, X_{[j]})\big|^\dagger_{(\hat{G}_{[j]}, \hat{X}_{[j]})} \hat{G}_{[j]} \geq 0 \tag{3.9}$$

Since the cost function $c_j(\cdot,\cdot)$ is defined to be convex and increasing in all arguments and evaluates to zero at the origin, we have that

$$\nabla_1\, c_j(G_{[j]}, X_{[j]})\big|^{\dagger}_{(\hat{G}_{[j]},\hat{X}_{[j]})}\, \hat{G}_{[j]} + \nabla_2\, c_j(G_{[j]}, X_{[j]})\big|^{\dagger}_{(\hat{G}_{[j]},\hat{X}_{[j]})}\, \hat{X}_{[j]} \geq c_j(\hat{G}_{[j]}, \hat{X}_{[j]}) \tag{3.10}$$

The result follows from the inequalities (3.9) and (3.10). □

Although Proposition 1 is similar to existing results regarding convex markets, it has important implications to our non-convex market pricing strategy. This is because the generator and commitment dispatch from Definition 1 is typically approximately equal to the generator and commitment values from the identified AIC KKT point from Definition 2 and as a result the profit resulting from the generation and commitment dispatch is approximately equal to the left-hand-side of (3.5). In fact, it is often the case that the generator and commitment values from the identified AIC KKT point for generator j will converge to the generator and commitment dispatch as ϵ is reduced to zero. This is stated in the following assumption.

Assumption 2. *For a specific generator j, the generation and commitment values associated with an AIC KKT point from Definition 2 approach the generation and commitment dispatch from Definition 1 as ϵ approaches zero.*

$$(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]}) \to (G^{\star}_{[j]}, R^{\star}_{[j]}, X^{\star}_{[j]}) \;\; \textit{as } \epsilon \to 0_+$$

There may be multiple generation and commitment values $(\hat{G}_{[j]}, \hat{R}_{[j]}, \hat{X}_{[j]})$ that satisfy the generalized KKT conditions in Definition 2 for a given value of ϵ. Assumption 2 does not assume that such generation and commitment values are unique. Rather, it intuitively states the following: Consider any sequence of generation and commitment values satisfying the generalized KKT conditions in Definition 2 where each iterate uses a different value of $\epsilon > 0$ such that ϵ approaches zero in limit. This sequence of generation and commitment values approaches the the generation and commitment dispatch $(G^{\star}_{[j]}, R^{\star}_{[j]}, X^{\star}_{[j]})$ in limit.

The following proposition states a key result indicating that a generator's profit approaches a non-negative quantity, representing the left-hand-side of (3.5), as ϵ approaches zero. This follows directly from continuity of the generator cost functions $c_j(G_{[j]}, X_{[j]})$, which were assumed continuous in their definition. To arrive at this proposition, the following assumption is additionally necessary, see Remark 3.3.

Assumption 3. *For a specific generator j located at bus i, the LIPs $(\hat{\Lambda}_{[i]}, \hat{\Pi}_{[i]})$ from Definition 2 are bounded.*

Proposition 2. *If generator j satisfies Assumptions 2 and 3, then its profits will approach the left-hand-side of* (3.5) *as ϵ approaches zero.*

$$\Big[M_j\hat{\Lambda}^{\dagger}\hat{R}_{[j]} + M_j\hat{\Pi}^{\dagger}\hat{G}_{[j]} - c_j(\hat{G}_{[j]}, \hat{X}_{[j]})\Big] - \Big[M_j\hat{\Lambda}^{\dagger}R^{\star}_{[j]} + M_j\hat{\Pi}^{\dagger}G^{\star}_{[j]} - c_j(G^{\star}_{[j]}, X^{\star}_{[j]})\Big] \to 0 \;\; \textit{as } \epsilon \to 0_+ \tag{3.11}$$

Sketch of Proof: The profit of generator j is expressed as a continuous function of the generation and commitment values because the function $c_j(\hat{G}_{[j]}, \hat{X}_{[j]})$ is defined to be continuous. The result then follows directly from Assumptions 2 and 3. □

Remark 3.3. Assumption 3 states that the prices from Definition 2 are bounded. Indeed, multiple prices may exist that satisfy the generalized KKT conditions in Definition 2. Furthermore, it is easy to construct an example with fixed demand such that real power prices are unbounded above. If these prices are unbounded above, then it is possible to construct a sequence of prices such that the generator payment increases to infinity and thus convergence of profit cannot be guaranteed. This is not a practical issue, but a theoretical one. Indeed, if there were multiple prices satisfying the KKT conditions such that they were unbounded above, the ISO surely wouldn't implement prices that were incredibly high and approaching infinity.

Proposition 2 states that any generator j that satisfies Assumptions 1, 2, and 3 will realize profits that approach a non-negative quantity as ϵ approaches zero. This result does not imply that generator j will realize non-negative profits. In fact, if ϵ is chosen to be very small, then it is possible for generator j to realize negative profits that are very small in magnitude. Indeed, this result instead implies that the lower bound on generator j's profits will be very close to zero if $\epsilon > 0$ is chosen to be very small.

Proposition 2 is a general result for any generator j that satisfies Assumptions 1, 2, and 3. It is possible that some generators satisfy these assumptions while others don't. For example, if there is flexible demand in the system, then Assumption 2 does not always hold as illustrated in the example in Section 4.1.3. When this happens such generators may realize a deficit when the market is cleared. On the other hand, such generators may instead represent the *incremental generator* that sets the price and still realize non-negative profit. Indeed, future work should further investigate the definition of an *incremental generator* and its relation to Assumption 2. The example in Section 4.3.2 illustrates the potential difficulty in defining an incremental generator. Specifically, this example shows that all generators may realize positive profits over the entire time horizon and that there may exist time intervals where no generator realizes exactly zero profit.

Assumption 2 does not necessarily hold in general and it may not be apparent that it typically holds for most generators. For this reason, we continue by providing two assumptions that ensure Assumption 2 holds for all generators. These two assumptions are sufficient to guarantee that Assumption 2 holds for all generators, but they are not necessary. Future work may focus on relaxing these assumptions.

As illustrated in the example in Section 4.1.3, Assumption 2 may not hold if there exists flexible demand. To eliminate this possibility, the following assumption ensures that no generator can consume real power. Recall that our definition of a generator was intentionally general to accommodate general market participants that may generate or consume energy. This assumption ensures that all real power demand is fixed.

Assumption 4. *No generator can consume real power:*

$$G_{[j]} \geq \mathbf{0} \quad \forall (G_{[j]}, R_{[j]}, X_{[j]}) \in \mathcal{X}_j \ \forall j \in [1, \gamma] \tag{3.12}$$

Assumption 4 along with the assumption that the transmission model is lossless is sufficient to prove that Assumption 2 holds for all generators. However, our formulation leaves the transmission model, represented by $\mathcal{T}_t$, general and is intended to accommodate the AC transmission model, which exhibits losses. For this reason, we use the following assumption that is more general than the assumption that the transmission model is lossless. Intuitively, this assumption states the following: for any net power injections $(P_t, Q_t) \in \mathcal{T}_t$ increasing the real power injection at bus i, denoted $P_{t,i}$, requires that the real power injection at some other bus k, denoted $P_{t,k}$ must decrease for the net power injections to remain feasible. This is trivially true for lossless networks because all net power injections $(P_t, Q_t) \in \mathcal{T}_t$ satisfy $P_t\mathbf{1} = 0$. We suggest that this assumption is likely to hold for practical AC transmission models. Intuitively, if this assumption does not hold, then there exists an operating point such that an increase in real power generation at a some bus results in an even larger increase in total transmission losses. To achieve such an operating point in a practical power system would likely require a very large amount of transmission losses that would cause transmission line thermal limits to be violated. Future work may attempt to soften this assumption.

Assumption 5. *Consider any two feasible net power injections* $(P'_t, Q'_t) \in \mathcal{T}_t$ *and* $(P''_t, Q''_t) \in \mathcal{T}_t$*. If* $P'_t \neq P''_t$ *then* $\exists i \in [1, n] : P'_{t,i} > P''_{t,i}$ *and* $\exists k \in [1, n] : P'_{t,k} < P''_{t,k}$*.*

The following proposition states that Assumption 2 holds if Assumptions 4 and 5 hold. As a result, by Propositions 1 and 2, any generators that additionally satisfy Assumptions 1 and 3 will see profits that approach a non-negative value as ϵ approaches zero.

Proposition 3. *Consider generation and commitment dispatch* $(G^\star, R^\star, X^\star)$ *from Definition 1 and the primal variables associated with an AIC KKT point* $(\hat{G}, \hat{R}, \hat{X}, \hat{P}, \hat{Q})$ *from Definition 2. If Assumptions 4 and 5 hold, then Assumption 2 holds for each generator* $j \in [1, \gamma]$*.*

Sketch of Proof: Define the feasible set of (3.2) in terms of ϵ as

$$\mathcal{F}(\epsilon) := \left\{(G, R, X, P, Q) \in \{\mathbb{R}^{\tau\times\gamma}\}^2 \times [\mathbf{0}, X^\star] \times \{\mathbb{R}^{\tau\times n}\}^2 : (3.2\text{a}) - (3.2\text{f})\right\}. \tag{3.13}$$

It is apparent that the generation and commitment dispatch is a feasible point in (3.2):

$$(G^\star, R^\star, X^\star, P^\star, Q^\star) \in \mathcal{F}(\epsilon) \quad \forall \epsilon \geq 0 \tag{3.14}$$

where $P^\star$ and $Q^\star$ follow from constraints (3.2a)-(3.2b). It is also apparent that $\mathcal{F}(\epsilon') \supseteq \mathcal{F}(\epsilon'')$ for any $\epsilon' > \epsilon'' > 0$.

We will now show that for any $(G', R', X', P', Q') \in \mathcal{F}(\epsilon')$ where $\epsilon' > 0$ and $G' \neq G^\star$ we can choose an $\epsilon'' > 0$ such that $\epsilon'' < \epsilon'$ where $(G', R', X', P', Q') \notin \mathcal{F}(\epsilon'')$. As a result, by the primal feasibility conditions in Definition 2, the real power generation values $\hat{G}$ associated with an AIC KKT point from Definition 2 approach the real power generation dispatch $G^\star$ from Definition 1 as ϵ approaches zero. Furthermore, in order for $\hat{G}$ to satisfy the upper bound in constraint (3.2e) along with the constraint $\hat{X} \in [\mathbf{0}, X^\star]$ the commitment values $\hat{X}$ must also approach the commitment dispatch $X^\star$ as ϵ approaches zero. Similarly, in order for $\hat{X}$ to satisfy the lower bound in constraint (3.2f) the reactive power values $\hat{R}$ must also approach the reactive power dispatch $R^\star$ as ϵ approaches zero. This leads to the desired result.

We have yet to show that for any $(G', R', X', P', Q') \in \mathcal{F}(\epsilon')$ where $\epsilon' > 0$ and $G' \neq G^\star$ we can choose an ϵ'' such that $0 < \epsilon'' < \epsilon'$ where $(G', R', X', P', Q') \notin \mathcal{F}(\epsilon'')$. We will show this by constructing an ϵ'' such that $(G', R', X', P', Q') \notin \mathcal{F}(\epsilon'')$. From $G' \neq G^\star$, we know that $\exists j \in [1, \gamma]$ and $t \in [0, \tau]$ where $G'_{t,j} \neq G^\star_{t,j}$. Let generator j be located at bus $i \in [1, n]$. We have two cases:

> *Case 1 ($G'_{t,j} > G^\star_{t,j}$):* Assumption 4 along with the upper bound in constraint (3.2e) and the constraint $X_{t,j} \in [0, X^\star_{t,j}]$ implies that $G'_{t,j} \leq G^\star_{t,j} + \epsilon$. This upper bound would be violated if ϵ'' is chosen such that $0 < \epsilon'' < G'_{t,j} - G^\star_{t,j}$. As a result, we would have $(G', R', X', P', Q') \notin \mathcal{F}(\epsilon'')$.
>
> *Case 2 ($G'_{t,j} < G^\star_{t,j}$):* By constraint (3.2a) either $P'_{t,i} > P^\star_{t,i}$ or $G'_{t,j'} > G^\star_{t,j'}$ for some other generator j' located at bus i. We analyze both cases separately:

Case 2.1 ($G'_{t,j'} > G^\star_{t,j'}$ for some generator $j' \neq j$ located at bus i): Assumption 4 along with the upper bound in constraint (3.2e) and the constraint $X_{t,j'} \in [0, X^\star_{t,j'}]$ implies that $G'_{t,j'} \leq G^\star_{t,j'} + \epsilon$. This upper bound would be violated if ϵ'' is chosen such that $0 < \epsilon'' < G'_{t,j'} - G^\star_{t,j'}$. As a result, we would have $(G', R', X', P', Q') \notin \mathcal{F}(\epsilon'')$.

Case 2.2 ($P'_{t,i} > P^\star_{t,i}$): By Assumption 5 there must exist bus $i' \in [1, n]$ where $P'_{t,i'} < P^\star_{t,i'}$. By constraint (3.2a) there must exist a generator k at bus i' where $G'_{t,k} > G^\star_{t,k}$. Assumption 4 along with the upper bound in constraint (3.2e) and the constraint $X_{t,k} \in [0, X^\star_{t,k}]$ implies that $G'_{t,k} \leq G^\star_{t,k} + \epsilon$. This upper bound would be violated if ϵ'' is chosen such that $0 < \epsilon'' < G'_{t,k} - G^\star_{t,k}$. As a result, we would have $(G', R', X', P', Q') \notin \mathcal{F}(\epsilon'')$. □

4 Simple Examples

This section presents simple and important examples that provide intuition behind AIC pricing. In this section we compare LIPs to LMPs using one and two bus examples and between one and three time intervals. This paper does not address computational effort and scalability of solving the AIC one-pass pricing problem and does not compare LIPs to CHPs, which is left for future work. Section 4.1 provides simple one-bus examples that only consider real power. Section 4.2 provides simple two-bus examples with one time interval that consider both real and reactive power. Section 4.3 provides examples that illustrate problem instances that exhibit degeneracy and produce multiple LIPs. Each subsection is further described as follows:

Section 4.1: Section 4.1.1 provides intuition using a simple one-interval example. Section 4.1.2 provides a one-interval example where a generator realizes a deficit because it does not satisfy Assumption 1. Section 4.1.3 provides a one-interval example that contains flexible demand and illustrates that flexible demand realizes a deficit because Assumption 4 is violated. Section 4.1.4 provides an example that builds intuition on a multi-interval example.

Section 4.2: Section 4.2.1 provides an example that illustrates the necessity of including the AC model of the transmission network in the ACUC problem by analyzing an infeasible dispatch produced by the DCUC problem. This example additionally illustrates that AIC pricing cannot capture reactive power phenomenon when using the DC model of the transmission network. Section 4.2.2 extends the previous example by illustrating the benefits of using the ACUC problem and AIC pricing. This example also shows that LIPs for reactive power may be significantly large, unlike LMPs for reactive power. Section 4.2.3 further extends the previous example by illustrating that the constraints that cause high incremental costs may not be binding in the pricing problem and thus may not be reflected in the prices. This is a potential shortfall of many different pricing structures including LMPs and LIPs. Section 4.2.4 provides an example that justifies the decision to price reactive power as a product in the market. In this example a generator is committed to provide only reactive power, causing it to incur start-up costs. Indeed, there is no way for this generator to recover these costs if reactive power is not priced as a product in the market.

Section 4.3: Section 4.3.1 provides an example where setting $\epsilon = 0$ results in multiple LIPs and setting $\epsilon > 0$ results in unique LIPs. To show this, the set of all possible LIPs are derived from the KKT conditions using a very simple one-bus one-time interval case. Section 4.3.1 extends the example from Section 4.1.4 by illustrating that multiple LIPs exist and different LIPs may provide different investment incentives. Furthermore, one choice of LIPs in this example results in all generators realizing positive profit over the time horizon.

4.1 One-Bus Real Power Examples

This subsection provides simple examples that only consider real power and one bus. Although we neglect reactive power, the resulting optimization problems are a special case of the ACUC problem (2.1) and AIC one-pass pricing problem (3.2), where reactive power is unconstrained by each of the constraints (2.1c) and (3.2c). Since reactive power is unconstrained, its price is zero.

Each case in this subsection uses the same two generators described in Table 1. The generators have a minimum and maximum real power output of 25p.u. and 100p.u. respectively. They both have linear cost functions with generator 1 having a marginal cost of 10\$/p.u. and generator 2 having a marginal cost of 5\$/p.u. Generator 2 additionally has very high start-up costs of \$5000. Generator 1 additionally has a minimum up time of 2 intervals. No ramping constraints are considered. Each example will further specify initial conditions for both generators. For each example we use $\epsilon = 10^{-5}$p.u.

Table 1: Parameters of both generators used in each example in Section 4.1.

	Min Output (p.u.)	Max Output (p.u.)	Marginal Cost (\$/p.u.)	Start-Up Cost (\$)	Min Up Time (# of intervals)
Gen 1	25	200	10	0	2
Gen 2	25	200	5	5000	1

4.1.1 One-Bus Example A

This example provides intuition and illustrates the benefits of AIC pricing. There is only one bus and one time interval, e.g. $n = 1$ and $\tau = 1$. The demand is fixed to 300p.u. and both generators are assumed to be initially uncommitted, meaning that their commitment status and generation values are zero at time $t = 0$. This implies that they must start-up and incur start-up costs in order to provide generation. This example results in LIPs of 30\$/p.u. and LMPs of 10\$/p.u. The optimal generation dispatch, total costs, and profits resulting from LMPs and LIPs are provided for both generators in Table 2.

Table 2: One-Bus Example A.

	Dispatch (p.u.)	Total Costs (\$)	LMP Profits (\$)	LIP Profits (\$)
Gen 1	100	1000	0	2000
Gen 2	200	6000	-4000	0

The optimal solution of the ACUC problem (2.1) has generator 1 producing 100p.u. at a cost of \$1000 and generator 2 producing 200p.u. at a cost of \$6000. The LMP is 10\$/p.u. and represents the marginal cost of serving load, which is 10\$/p.u. because generator 1 is the marginal generator and has a marginal cost of 10\$/p.u. The LMP results in generator 1 realizing zero profits and generator 2 realizing a deficit of \$4000. The LIP is 30\$/p.u. and represents the total costs of generator 2 averaged over its generation dispatch, e.g. $\frac{\$6000}{30\text{p.u.}}$. This results in generator 1 realizing \$2000 profits and generator 2 realizing 0 profits. In this context, we say that generator 2 is the incremental generator and generator 1 is an infra-incremental generator.

In this example the LMPs result in both generators realizing non-positive profit and thus do not provide sufficient profit to cover the future generation capacity investment necessary to attain a sustainable electricity market. In contrast to LMPs, LIPs provide better investment incentives. The additional profits accrued by generator 2 when using LIPs should stimulate investment in future generation.

4.1.2 One-Bus Example B

This example extends Example A by reducing the demand to 150p.u. and changing the initial conditions such that generator 1 no longer satisfies Assumption 1. Indeed, we see that generator 1 realizes a deficit even when using AIC pricing. Specifically, both generators are assumed to be initially committed, meaning that their commitment statuses are one at time $t = 0$ and so generator 2 does not incur start-up costs over the optimization horizon. Furthermore, generator 1 is assumed to have started-up in the previous time interval and so it must be committed during the optimized time interval in order to satisfy its minimum up time requirement. This means that generator 1 must remain committed and so it violates Assumption 1 because $(0, 0, 0) \notin \mathcal{X}_1$.

This example results in the same LIP and LMP of 5\$/p.u. The optimal generation dispatch, total costs, and profits resulting from LMP and LIP are provided for both generators in Table 3. The optimal solution of the ACUC problem (2.1) has generator 1 producing 25p.u. at a cost of \$250 and generator 2 producing 125p.u. at a cost of \$625. Generator 1 realizes a deficit of \$125 using either LMP or LIP and generator 2 realizes a profit of zero.

Table 3: One-Bus Example B.

	Dispatch (p.u.)	Total Costs (\$)	LMP Profits (\$)	LIP Profits (\$)
Gen 1	25	250	-125	-125
Gen 2	125	625	0	0

In this example, the reason generator 1 realizes a deficit when using LIPs is because the commitment decision that caused the deficit was made prior to the present time of $t = 1$. Indeed, this problem could have been corrected by clearing the market at time $t = 0$ using an ACUC problem and an AIC one-pass pricing problem that spanned two time intervals of $t = 0$ and $t = 1$ assuming that all uncertainty is accurately forecast at both time intervals. In this case the start-up costs of generator 1 would have been captured in the LIPs. However, in practice, forecasting errors along with complex rolling horizon implementations of the ACUC problem may cause Assumption 1 to be occasionally violated and as a result some generators could realize a deficit. Future work may focus on developing effective rolling horizon implementations of the ACUC problem and the AIC one-pass pricing problem. It is also important to recognize that the same issues occur with other pricing structures. Indeed, in this example, the LMP also results in generator 1 realizing a deficit.

4.1.3 One-Bus Example C

This example extends Example A by introducing flexible demand in addition to the already existing fixed demand of 300p.u. Flexible demand is modeled as a generator with negative generation (representing demand). The parameters of flexible demand are shown in Table 4. Indeed, Assumption 4 is violated because flexible demand is modeled as a generator with negative output.

Table 4: Adding Flexible Demand (Modeled as as Generator 3).

	Min Output (p.u.)	Max Output (p.u.)	Marginal Cost ($/p.u.)	Start-Up Cost ($)	Min Up Time (# of intervals)
Flex Demand (Gen 3)	-10	0	20	0	0

The optimal dispatch, representing the solution of the ACUC problem (2.1), is provided in Table 5. The prices remain the same as in Example A, namely, the LMPs are 10$/p.u. and the LIPs are 30$/p.u. In this case, the dispatch from the AIC one-pass pricing problem (3.2) approaches the value of $\hat{G} = [110, 190, 0]$ as ϵ approaches zero, which deviates from the optimal dispatch of $G^{\star} = [110, 200, -10]$. This means that Assumption 2 is violated and Proposition 2 does not apply. Indeed, we see that flexible demand, which is modeled as generator 3, realizes negative profits of -$100.

Table 5: One-Bus Example C.

	Dispatch (p.u.)	Total Costs ($)	LMP Profits ($)	AIC Profits ($)
Gen 1	110	1100	0	2200
Gen 2	200	-200	-4000	0
Flex Demand (Gen 3)	-10	-200	100	-100

In this example, the ACUC problem commits generator 2 in order to satisfy the fixed demand. After this generator has been committed, the marginal cost of serving an additional small amount of demand drops below the marginal utility of the flexible demand and so the flexible demand is dispatched upward. This is the cause of the violation of Assumption 2. For larger systems, this type of problem is not likely to occur for a significant number of flexible demand market participants. For example, we can imagine that the fixed demand of 300p.u. might represent a combination of many different market participants representing demand with very high utility, in which case those market participants would still be realizing a profit due to their high utility.

4.1.4 One-Bus Example D

This example extends Example A by introducing multiple time intervals. We consider three time intervals where the demand is fixed to $[30, 230, 230]$ in units of p.u. Both generators have initial commitment statuses of zero during time interval $t = 0$ and must start-up in order to provide generation. Table 6 provides the solution and profits for each generator. Using the dual simplex method in Gurobi to solve (3.2), we obtain the LMPs $[5, 10, 10]$ and the LIPs $\hat{\Pi}^{\dagger}_{[1]} = [171.67, 10, 10]$ in units of $/p.u. Note: Section 4.3.2 elaborates on this example by showing that there is degeneracy and multiple LIPs exist, some of which provide better investment signals to increase capacity during peak demand.

The LIPs $\hat{\Pi}^{\dagger}_{[1]} = [171.67, 10, 10]$ are very intuitive because each interval has a corresponding *incremental generator* and the LIP for each interval represents the total costs incurred by the incremental generator divided by its real power output. For example, Generator 2 is the incremental generator for time interval 1 and incurs total costs of \$5150 due to its start-up costs. The LIP during time interval 1 represents generator 2's costs divided by its real power output: $\hat{\Pi}^{\dagger}_{1,1} = \frac{\$5150}{30\text{p.u.}} = 171.67\$/\text{p.u.}$ Similarly, the LIPs in time intervals 2 and 3 can be computed by generator 1, which is the incremental generator for these time intervals. Furthermore, these LIPs result in all generators realizing non-negative profit during each time interval and the incremental generator realizes exactly zero profits during the interval in which it is the incremental generator. The profits of generator 1 by interval are $[0, 1000, 1000]$ and the profits of generator 2 by interval are $[0, 0, 0]$ in units of dollars.

Table 6: One-Bus Example D. ($\hat{\Pi}^{\dagger}_{[1]} = [171.67, 10, 10]$ in units of \$/p.u.)

	Dispatch $G^{\star\dagger}_{[j]}$ (p.u.)	Commitment $X^{\star\dagger}_{[j]}$	Costs per Interval (\$)	Total Costs (\$)	LMP Profits (\$)	LIP Profits (\$)
Gen 1	[0,30,30]	[0,1,1]	[0,300,300]	600	0	0
Gen 2	[30,200,200]	[1,1,1]	[5150,1000,1000]	7150	-3000	2000

4.2 Two-Bus Examples

This section provides simple examples with two busses and one time interval that represent variant's of the test case in Figure 1. The examples will illustrate the deficiencies of the DC model of the transmission system and will illustrate the impact of reactive power on AIC pricing by examining cases where the LMPs result in generators realizing a deficit after the market is cleared. The variations to the test case in Figure 1 will be stated explicitly in each subsection. For each example we use $\epsilon = 10^{-5}$p.u.

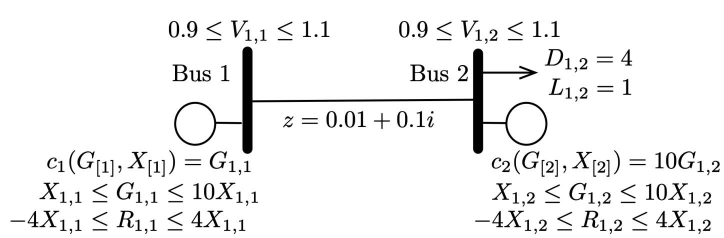

Figure 1: Simple two-bus AC transmission test case.

In this example the demand is located at bus 2. The remote generation (at bus 1) is less expensive and the local generation is more expensive. The private constraints for both generators are the same. We will write them explicitly for both generators so we can reference the individual constraints throughout this section. Consistent with previously defined notation, we use $G_{[1]}$ to represent the first column of the matrix G; however, in this example, G is a 1 by 2 matrix, so $G_{[1]}$ is scalar.

<u>Generator 1</u>

$$c_1(G_{[1]}, X_{[1]}) = G_{1,1} \tag{C1}$$

$$\mathcal{X}_1 = \{ (G_{[1]}, R_{[1]}, X_{[1]}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : \tag{G1}$$

$$X_{1,1} \le G_{1,1} \le 10X_{1,1} \tag{G1.1}$$

$$-4X_{1,1} \le R_{1,1} \le 4X_{1,1} \} \tag{G1.2}$$

<u>Generator 2</u>

$$c_2(G_{[2]}, X_{[2]}) = 10G_{1,2} \tag{C2}$$

$$\mathcal{X}_2 = \{ (G_{[2]}, R_{[2]}, X_{[2]}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : \tag{G2}$$

$$X_{1,2} \le G_{1,2} \le 10X_{1,2} \tag{G2.1}$$

$$-4X_{1,2} \le R_{1,2} \le 4X_{1,2} \} \tag{G2.2}$$

In this section we focus on the AC transmission model of the two-bus test case; however, we also compare it to the DC transmission model. The feasible set of net power injections are defined below for both models. The DC transmission model does not place constraints on the reactive power and does not consider voltage magnitudes. The conductance of the line is represented by $g = \frac{0.01}{0.01^2+0.1^2}$ and the susceptance of the line is represented by $b = \frac{0.1}{0.01^2+0.1^2}$, where all units are in *per unit*. The voltage angle at bus 1 is denoted Δ and bus 2 is the reference bus.

<u>AC Transmission Model</u>

$$\mathcal{T}_1 = \{ (P_1, Q_1) \in \mathbb{R}^2 \times \mathbb{R}^2 : \tag{AC}$$

$$\exists (V, \Delta) \in \mathbb{R}^2 \times \mathbb{R} \text{ where}$$

$$0.9 \le V_{1,1} \le 1.1 \tag{AC.1}$$

$$0.9 \le V_{1,2} \le 1.1 \tag{AC.2}$$

$$P_{1,1} = gV_{1,1}^2 - gV_{1,1}V_{1,2}\cos\Delta - bV_{1,1}V_{1,2}\sin\Delta \tag{AC.3}$$

$$P_{1,2} = gV_{1,2}^2 - gV_{1,1}V_{1,2}\cos\Delta + bV_{1,1}V_{1,2}\sin\Delta \tag{AC.4}$$

$$Q_{1,1} = -bV_{1,1}^2 - gV_{1,1}V_{1,2}\sin\Delta + bV_{1,1}V_{1,2}\cos\Delta \tag{AC.5}$$

$$Q_{1,2} = -bV_{1,2}^2 + gV_{1,1}V_{1,2}\sin\Delta + bV_{1,1}V_{1,2}\cos\Delta \} \tag{AC.6}$$

<u>DC Transmission Model</u>

$$\mathcal{T}_1 = \{ (P_1, Q_1) \in \mathbb{R}^2 \times \mathbb{R}^2 : \tag{DC}$$

$$\exists \Delta \in \mathbb{R} \text{ where}$$

$$P_{1,1} = -b\sin\Delta \tag{DC.1}$$

$$P_{1,2} = b\sin\Delta \} \tag{DC.2}$$

4.2.1 Two-Bus DC Example

This example uses the DC model of the transmission system, which does not enforce voltage magnitude or reactive power constraints. The dispatch and the prices are provided in Table 7. We see that all demand is served by generator 1 and there is no need to commit generator 2, which is more expensive. LMPs are identical to LIPs and both generators realize zero profits.

The commitment and dispatch resulting from the DC transmission model is not feasible for the AC transmission model. In fact, if generator 2 stays uncommitted, then generator 1 would need to produce real power of 4.24p.u. and reactive power of 3.43p.u. to satisfy the demand; however, this would result in a bus 2 voltage of 0.82p.u., which violates the lower voltage limit constraint. To attain a feasible dispatch from the DC solution, fast acting corrective action would need to be taken. The cost of such corrective action can be lower bounded by the increase in cost of the UC

Table 7: Two-bus DC example.

	Dispatch				LMP			LIP		
	Real (p.u.)	Reactive (p.u.)	Commitment	Costs ($)	Real $(\frac{\$}{\text{p.u.}})$	Reactive $(\frac{\$}{\text{p.u.}})$	Profits ($)	Real $(\frac{\$}{\text{p.u.}})$	Reactive $(\frac{\$}{\text{p.u.}})$	Profits ($)
Gen 1	4	N/A	1	4	1	N/A	0	1	N/A	0
Gen 2	0	N/A	0	0	1	N/A	0	1	N/A	0

problem when considering the AC transmission model, as shown in the next section. The ideal corrective action would result in the optimal dispatch obtained by the ACUC problem with the AC transmission model, which is provided in Table 8. We emphasize that this is an ideal corrective action because such practical corrective action would likely require deployment of costly fast-acting resources. Table 8 illustrates that generator 2 ultimately realizes a deficit of $9 when using either LMPs or LIPs. In this case, the ISO would provide make whole payments to generator 2 for it to realize a profit of zero.

Table 8: Two-bus DC example after ideal corrective action (best possible corrective action).

	Dispatch After Corrective Action				LMP			LIP		
	Real (p.u.)	Reactive (p.u.)	Commitment	Costs ($)	Real $(\frac{\$}{\text{p.u.}})$	Reactive $(\frac{\$}{\text{p.u.}})$	Profits ($)	Real $(\frac{\$}{\text{p.u.}})$	Reactive $(\frac{\$}{\text{p.u.}})$	Profits ($)
Gen 1	3.08	0.09	1	3.08	1	N/A	0	1	N/A	0
Gen 2	1	1.70	1	10	1	N/A	-9	1	N/A	-9

4.2.2 Two-Bus AC Example A

This example uses the AC model of the transmission system. The simplicity of the two-bus problem allows the ACUC problem to be solved to optimality by evaluating the objective for each combination of possible commitment statuses. We find that generator 2 must be committed to satisfy the voltage magnitude limits. Furthermore, it is optimal for both generators to be committed because generator 1 is much less expensive than generator 2. Table 9 provides results.

Table 9: Two-bus AC example.

	Dispatch				LMP			LIP		
	Real (p.u.)	Reactive (p.u.)	Commitment	Costs ($)	Real $(\frac{\$}{\text{p.u.}})$	Reactive $(\frac{\$}{\text{p.u.}})$	Profits ($)	Real $(\frac{\$}{\text{p.u.}})$	Reactive $(\frac{\$}{\text{p.u.}})$	Profits ($)
Gen 1	3.08	0.09	1	3.08	1	0	0	1.01	3.94	0.36
Gen 2	1	1.70	1	10	1.05	0	-8.95	3.21	4.01	0

Unlike the previous example that used the DC transmission model, this example uses the AC transmission model and so the ACUC problem (2.1) is able to determine an AC feasible commitment and dispatch. However, LMPs cause generator 2 to realize a deficit of $8.95 because it operates at its minimum real power output level of 1p.u. and has a marginal cost higher than the marginal generator (generator 1). On the other hand, LIPs that consider the full AC transmission model allow generator 2 to receive payment equal to its cost.

LIPs provide improved investment signals as compared to LMPs. LIPs more accurately capture the costs of generation and generally indicate optimal locations for generation investment. Notice that the LIPs for real and reactive power are elevated at bus 2, where the generation is needed. Indeed, both prices are elevated at bus 2, indicating that investment in real or reactive power could help reduced costs. It is not immediately clear why the reactive power versus the real power price at bus 1 is elevated. Bus 1 has an excess of both real and reactive power, so it seems arbitrary that the reactive power price would be higher in this case.

4.2.3 Two-Bus AC Example B

This subsection illustrates that it is difficult to determine the "cause" of generator 2's commitment and that LIPs generally do not reflect or price the constraints that cause generator 2 to commit. To emphasize this point, many different variations of the test case in Figure 1 are provided in Table 10 that each produce the same dispatch and prices shown in Table 9. Each case relaxes the lower bound on the voltage magnitude at bus 2, which alone would allow generator 2 to remain uncommitted. However, each case introduces a more restrictive constraint that again causes generator 2 to commit. In Case 1, generator 2 commits due to a deficiency of real power at bus 1. In Case 2, generator 2 commits due to a deficiency of reactive power at bus 1. However, both Case 1 and Case 2 result in the same LIPs and thus LIPs do not accurately provide incentive for real and reactive power investment at bus 1. This reveals a limitation of AIC pricing. In Case 3, generator 2 commits due to a line limit constraint that couples real and reactive power. Similarly, this constraint is not reflected in the LIPs because it is not binding at the optimal solution of the AIC one-pass pricing problem (3.2). Indeed, this limitation of the AIC pricing structure is also a limitation of other pricing structures, such as the marginal pricing structure.

Table 10: Variations of the two-bus AC example.

Case 1	Case 2	Case 3
Replace (AC.2) with	Replace (AC.2) with	Replace (AC.2) with
$0.8 \leq V_{1,2} \leq 1.1$	$0.8 \leq V_{1,2} \leq 1.1$	$0.8 \leq V_{1,2} \leq 1.1$
Replace (G1.1) with	Replace (G1.2) with	Add line limit constraint to $\mathcal{T}_1$
$X_{1,1} \leq G_{1,1} \leq 4X_{1,1}$	$-4X_{1,1} \leq R_{1,1} \leq 3X_{1,1}$	$\sqrt{P_{1,2}^2 + Q_{1,2}^2} \leq 3X_{1,1}$

4.2.4 Two-Bus AC Example C

This example changes the parameters of generator 2 so that its minimum real power output is reduced to zero and it realizes high start-up costs. In other words, constraint (G2.1) is replaced with $0 \leq G_{1,2} \leq 10X_{1,2}$ and the function (C2) is replaced with $c_2(G_{[2]}, X_{[2]}) = 10G_{1,2} + 10X_{1,2}$. Once again, in this example generator 2 must be committed to satisfy the voltage magnitude constraints. Table 11 provides the results. The optimal dispatch commits generator 2 to provide only reactive power due to its high real power generation costs. Generator 2 incurs \$10 in start-up costs but receives zero payments when using LMPs because the reactive power LMP is zero. LIPs correct for this problem by increasing the reactive power price to a value where generator 2 recovers its cost and realizes zero profit.

This example illustrates why reactive power needs to be priced as a product in the electricity market. If real power were the only market product in this example, then generator 2 would not be able to receive payment at its optimal real power dispatch of $P^{\star}_{[2]} = 0$ and would necessarily realize a deficit.

Table 11: Two-Bus AC Example C.

	Dispatch				LMP			LIP		
	Real (p.u.)	Reactive (p.u.)	Commitment	Costs ($)	Real $(\frac{\$}{\text{p.u.}})$	Reactive $(\frac{\$}{\text{p.u.}})$	Profits ($)	Real $(\frac{\$}{\text{p.u.}})$	Reactive $(\frac{\$}{\text{p.u.}})$	Profits ($)
Gen 1	4.14	0.31	1	4.14	1	0	0	1.01	4.46	1.36
Gen 2	0	2.12	1	10	1.07	0	-10	4.54	4.72	0

4.3 Examples with Multiple Locational Incremental Prices

This section presents examples that experience degeneracy and multiple LIPs exist that satisfy the conditions in Definition 2. The first example results in multiple LIPs if $\epsilon = 0$ and results in unique LIPs when $1 \gg \epsilon > 0$. This example justifies the inclusion of the ϵ parameter in the AIC one-pass pricing problem (3.2). The second example examines a case where a generator must be committed to accommodate the peak load, but is committed earlier than the occurance of the peak load because it has low marginal cost. In this example there exist multiple LIPs that allocate the incremental costs differently among the time intervals. This suggests important questions to be addressed in future work: How can we characterize all possible LIPs? How do we define the incremental generator? When there are multiple LIPs, which should be chosen to represent the prices?

4.3.1 Multiple Price Example A

This example illustrates that setting $\epsilon = 0$ may cause degeneracy and result in multiple LIPs. Furthermore, setting $\epsilon > 0$ can eliminate degeneracy and result in unique LIPs. Indeed, in practice it is difficult to analyze degeneracy using off-the-shelf optimization solvers because they specialize in determining one optimal solution to an optimization problem and do not provide features that characterize all optimal solutions. For this reason we focus on a simple example where a closed-form solution of the LIPs can be derived from the KKT conditions.

The test case includes one bus, one time interval, and one generator. The generator costs are $c_1(G_{1,1}, X_{1,1}) = G_{1,1}^2$ and the generator is left unconstrained as it does not have a maximum or minimum generation limit. The fixed demand in the system is $D_{1,1}$. The only feasible point and thus the optimal solution is $(G^{\star}_{1,1}, X^{\star}_{1,1}) = (D_{1,1}, 1)$. The resulting AIC one-pass pricing problem is as follows:

$$\min_{G_{1,1}\in\mathbb{R}, X_{1,1}\in\mathbb{R}} \quad G_{1,1}^2 \tag{4.15}$$

$$(\Pi_{1,1}) \qquad G_{1,1} = D_{1,1} \tag{4.15a}$$

$$(\underline{\Gamma}_{1,1}, \overline{\Gamma}_{1,1}) \qquad \left(G^\star_{1,1} - \epsilon\right) X_{1,1} \le G_{1,1} \le \left(G^\star_{1,1} + \epsilon\right) X_{1,1} \tag{4.15b}$$

$$(\underline{\Psi}_{1,1}, \overline{\Psi}_{1,1}) \qquad 0 \le X_{1,1} \le 1 \tag{4.15c}$$

where the interval constraints on the commitment variables, which are imposed implicitly in (3.2), are represented explicitly in (4.15c) and are assigned Lagrange multipliers $\underline{\Psi} \in \mathbb{R}$ and $\overline{\Psi} \in \mathbb{R}$. Similarly, constraints (4.15b) are assigned the Lagrange multipliers $\underline{\Gamma} \in \mathbb{R}$ and $\overline{\Gamma} \in \mathbb{R}$.

In this example the traditional KKT conditions are simple enough to derive closed form expressions of the LIPs $\hat{\Pi}_{1,1}$. The KKT conditions represent *primal feasibility*, indicating that (4.15a)-(4.15c) are satisfied, along with the following *stationarity*, *complimentary slackness*, and *dual feasibility* conditions.

Stationarity

$$0 = 2\hat{G}_{1,1} - \hat{\Pi}_{1,1} - \hat{\underline{\Gamma}}_{1,1} + \hat{\overline{\Gamma}}_{1,1} \tag{4.16}$$

$$0 = \hat{\overline{\Psi}}_{1,1} - \hat{\underline{\Psi}}_{1,1} + \hat{\underline{\Gamma}}_{1,1}(G^\star_{1,1} - \epsilon) - \hat{\overline{\Gamma}}_{1,1}(G^\star_{1,1} + \epsilon) \tag{4.17}$$

Complimentary Slackness

$$0 = \hat{\underline{\Gamma}}_{1,1}((G^\star_{1,1} - \epsilon)\hat{X}_{1,1} - \hat{G}_{1,1}) \tag{4.18}$$

$$0 = \hat{\overline{\Gamma}}_{1,1}(\hat{G}_{1,1} - (G^\star_{1,1} + \epsilon)\hat{X}_{1,1}) \tag{4.19}$$

$$0 = -\hat{\underline{\Psi}}_{1,1}\hat{X}_{1,1} \tag{4.20}$$

$$0 = \hat{\overline{\Psi}}_{1,1}(\hat{X}_{1,1} - 1) \tag{4.21}$$

Dual Feasibility

$$0 \le \hat{\underline{\Gamma}}_{1,1} \quad 0 \le \hat{\overline{\Gamma}}_{1,1} \tag{4.22}$$

$$0 \le \hat{\underline{\Psi}}_{1,1} \quad 0 \le \hat{\overline{\Psi}}_{1,1} \tag{4.23}$$

When setting $\epsilon = 0$ the LIP can attain any value $\hat{\Pi}_{1,1} \ge 2\hat{G}_{1,1}$ and so the LIP is not unique. We continue to show this explicitly: There is only one feasible solution and the primal feasibility conditions require the optimal generation and commitment dispatch to be $(\hat{G}_{1,1}, \hat{X}_{1,1}) = (D_{1,1}, 1)$. As a result, the complimentary slackness conditions (4.18), (4.19), and (4.21) hold for all values of Lagrange multipliers. The complimentary slackness condition (4.20) implies $\hat{\underline{\Psi}}_{1,1} = 0$. Then the stationarity condition (4.17) implies $\hat{\underline{\Gamma}}_{1,1} - \hat{\overline{\Gamma}}_{1,1} = -\frac{\hat{\overline{\Psi}}_{1,1}}{G^\star_{1,1}}$. This expression can be substituted into the stationarity condition (4.16) to obtain an expression for the LIP: $\hat{\Pi}_{1,1} = 2\hat{G}_{1,1} + \frac{\hat{\overline{\Psi}}_{1,1}}{G^\star_{1,1}}$. Since, the Lagrange multiplier $\hat{\overline{\Psi}}_{1,1}$ is only constrained by the inequality from (4.23), the LIP can be any value such that $\hat{\Pi}_{1,1} \ge 2\hat{G}_{1,1}$.

When setting $1 \gg \epsilon > 0$ the LIP is unique and is expressed as $\hat{\Pi}_{1,1} = 2\hat{G}_{1,1}$. We continue to show this explicitly: By (4.15a) there is only one feasible generation value $\hat{G}_{1,1} = D_{1,1}$; however, there are multiple optimal commitment values. Specifically, by the upper bounds in (4.15b) and (4.15c), the commitment variable can take on any value $\frac{D_{1,1}}{D_{1,1}+\epsilon} \le \hat{X}_{1,1} \le 1$ without impacting the objective value. From (4.16) the LIP is expressed as $\hat{\Pi}_{1,1} = 2\hat{G}_{1,1} - \hat{\underline{\Gamma}}_{1,1} + \hat{\overline{\Gamma}}_{1,1}$. We will arrive at the desired result by showing that $\hat{\underline{\Gamma}}_{1,1} = 0$ and $\hat{\overline{\Gamma}}_{1,1} = 0$. We will show this in two cases. In case 1 the commitment values satisfy $\frac{D_{1,1}}{D_{1,1}+\epsilon} < \hat{X}_{1,1} \le 1$. In this case the upper and lower bounds in (4.15b) are not binding and thus complimentary slackness constraints (4.18) and (4.19) directly imply that $\hat{\underline{\Gamma}}_{1,1} = 0$ and $\hat{\overline{\Gamma}}_{1,1} = 0$. In case 2 the commitment values satisfy $\frac{D_{1,1}}{D_{1,1}+\epsilon} = \hat{X}_{1,1}$. In this case the lower bound in (4.15b) is not binding and thus complimentary slackness constraints (4.18) directly imply that $\hat{\underline{\Gamma}}_{1,1} = 0$. Furthermore, the upper and lower bounds in (4.15b) are not binding and thus complimentary slackness constraints (4.20) and (4.21) directly imply that $\hat{\underline{\Psi}}_{1,1} = 0$ and

$\hat{\hat{\Psi}}_{1,1} = 0$. As a result, the stationarity condition (4.17) implies that $\hat{\hat{\Gamma}}_{1,1}(G^\star_{1,1} + \epsilon) = 0$ and thus $\hat{\hat{\Gamma}}_{1,1} = 0$.

4.3.2 Multiple Price Example B

This section extends the example from Section 4.1.4 by illustrating that multiple LIPs exist due to degeneracy. This degeneracy occurs even when the parameter ϵ is chosen to be greater than zero, as it is set to $\epsilon = 10^{-5}$p.u. in this example. As presented in Section 4.1.4, Gurobi's dual simplex method produces the LIPs $\hat{\Pi}^\dagger_{[1]} = [171.67, 10, 10]$. However, Gurobi's barrier method produces the LIPs $\hat{\Pi}^\dagger_{[1]} = [5, 10, 25]$ and the default settings in IPOPT produce the LIPs $\hat{\Pi}^\dagger_{[1]} = [5.07, 18.21, 16.82]$. Furthermore, small changes in the AIC one-pass problem can result in significant changes the the LIPs. For example, if we use $\epsilon = 10^{-4}$, then Gurobi's dual simplex method produces the LIPs $\hat{\Pi}^\dagger_{[1]} = [5, 30, 10]$. All prices are in units of \$/p.u.

In this case it is not clear which LIPs are the best. Although the LIPs $\hat{\Pi}^\dagger_{[1]} = [171.67, 10, 10]$ are intuitive based on the simple calculations involving the incremental generator from Section 4.1.4, these prices may not provide accurate incentives for investment. Indeed, due to the high price in time interval 1, these prices provide incentive for generation capacity investment in time interval 1. However, demand is very low during this time interval and generation capacity investment would be more appropriate for time intervals 2 and 3.

In this example, both generators must be committed in order to satisfy the high demand in time intervals $t = 2$ and $t = 3$. This means that the start-up costs of generator 2 will inevitably be incurred. Recognizing this, the ACUC problem (2.1) decides to commit generator 2 before it is necessary, in time interval $t = 1$, in order to take advantage of its low marginal costs. As a result, the high demand in time intervals $t = 2$ and $t = 3$ cause the start-up costs to be incurred in time interval $t = 1$. For this reason, it might make sense to choose LIPs that have high prices during time intervals 2 and 3, providing incentive for generation investment during these time intervals or demand shifting away from these time intervals.

Let's further analyze the LIPs $\hat{\Pi}^\dagger_{[1]} = [5, 30, 10]$, which were determined with $\epsilon = 10^{-4}$ and using Gurobi's dual simplex method. Interestingly, these LIPs effectively allocate these incremental costs to a time interval that *causes* the commitment to occur, $t = 2$, instead of the time interval that the costs actually occur, $t = 1$. Indeed, the LIP for time interval $t = 2$ represents the sum of generator 2's start-up costs of \$5000 incurred during time interval $t = 1$ and generator 2's variable costs of \$1000 incurred during time interval $t = 2$ averaged over generator 2's power output during time interval $t = 2$, e.g. $\frac{\$6000}{200\text{p.u.}} = 30\frac{\$}{\text{p.u.}}$. Table 12 provides results using these LIPs.

Table 12: Multiple Price Example B. ($\hat{\Pi}^\dagger_{[1]} = [5, 30, 10]$ in units of \$/p.u.)

	Dispatch $G^\dagger_{[j]}$ (p.u.)	Commitment $X^\dagger_{[j]}$	Costs per Interval (\$)	Total Costs (\$)	LMP Profits (\$)	LIP Profits (\$)
Gen 1	[0,30,30]	[0,1,1]	[0,300,300]	600	0	600
Gen 2	[30,200,200]	[1,1,1]	[5150,1000,1000]	7150	-3000	1000

The LIPs $\hat{\Pi}^{\dagger}_{[1]} = [5, 30, 10]$ additionally illustrate two important things. First, it is possible for all generators to realize positive profits throughout the time horizon as generator 1 realizes a profit of \$600 and generator 2 realizes a profit of \$1000. Second, given the LIPs it is not always easy to determine the incremental generator for a given time interval because there may exist time intervals where no generator realizes exactly zero profit. Indeed, the profits for each time interval for generators 1 and 2 are $[0, 600, 0]$ and $[-5000, 5000, 1000]$ respectively in units of dollars. Notice that no generator during time interval 2 realizes exactly zero profits. Also, notice that generator 2 realizes negative profits during time interval 1, but recovers those lost profits during time interval 2.

5 Conclusion

This paper proposed an AIC pricing structure for the ACUC problem in the context of the day-ahead electricity market. An AIC one-pass pricing problem was proposed that represents a continuously constrained variant of the ACUC problem and LIPs were defined by the local optimal Lagrange multipliers of the power balance constraints. To avoid degeneracy, the AIC one-pass pricing problem introduces a small $\epsilon > 0$ parameter that relaxes some constraints. Theoretical results were informally proven regarding the profitability of market participants. Under certain assumptions generators are shown to have profit that converges to a non-negative value as $\epsilon > 0$ approaches zero. Simple examples then provided insights into the profitability results, justified the necessity of AC transmission considerations, and indicated many directions for future work.

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