GravitySDP: A Solver for Sparse Mixed-Integer Semidefinite Programming

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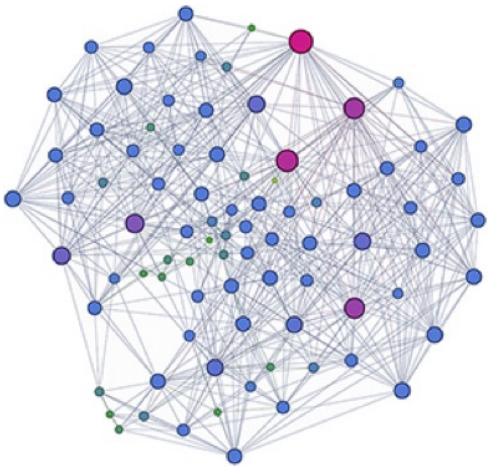
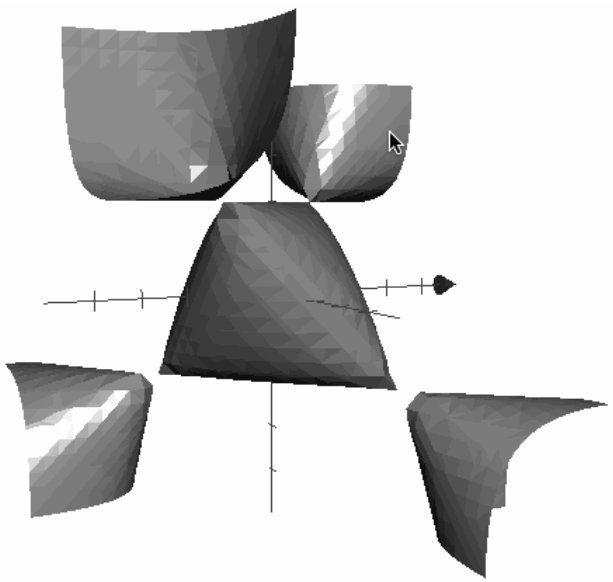
INFORMS 23





LA-UR-23-25819

The story begins in 2018

SDP relaxation for the ACOPF

	Full-SDP (Mosek)		Sparse-SDP (Mosek)		Sparse-SDP (SDPT3)		3D-Determinant Cuts (Ipopt)	
Instance	Gap	Time	Gap	Time	Gap	Time	Gap	Time
5_pjm	5.2%	0.0	5.2%	0.1	5.22%	0.66	5.22%	**0.04**
118_ieee	ERR	319.1	0.2%	1.3	0.18%	3.50	0.20%	**0.84**
162_ieee_dtc	MEM		2.4%	8.5	2.26%	10.70	2.36%	**5.48**
240_pserc	MEM		2.3%	3.5	ERR	ERR	2.30%	**4.39**
300_ieee	MEM		0.4%	5.4	0.11%	6.80	0.14%	**2.09**
3_lmbd__api	5.0%	0.0	5.0%	0.0	7.34%	0.50	4.99%	**0.02**
24_ieee_rts__api	ERR	ERR	ERR	ERR	2.06%	1.20	2.09%	**0.06**
30_as__api	ERR	ERR	ERR	ERR	ERR	ERR	7.18%	**0.11**
30_fsr__api	0.3%	2.9	0.3%	0.2	0.28%	1.20	0.81%	**0.11**
39_epri__api	0.2%	11.6	0.3%	0.3	0.18%	1.30	0.22%	**0.19**
73_ieee_rts__api	ERR	ERR	ERR	ERR	2.91%	2.60	2.97%	**0.28**
89_pegase__api	ERR	ERR	ERR	ERR	6.88%	**5.40**	6.96%	8.03
118_ieee__api	ERR	ERR	11.2%	1.4	11.14%	4.60	11.78%	**0.85**
162_ieee_dtc__api	MEM		1.7%	8.7	1.71%	10.70	1.81%	**6.89**
24_ieee_rts__sad	ERR	ERR	ERR	ERR	4.36%	1.30	2.53%	**0.06**
57_ieee__sad	0.1%	40.9	0.1%	0.6	0.16%	1.20	0.15%	**0.17**
73_ieee_rts__sad	ERR	ERR	ERR	ERR	2.75%	1.90	1.58%	**0.31**
118_ieee__sad	ERR	ERR	3.7%	1.4	5.45%	3.90	3.93%	**0.72**
162_ieee_dtc__sad	MEM		2.4%	**8.7**	2.67%	10.70	2.4%	10.00
240_pserc__sad	MEM		ERR	ERR	ERR	ERR	4.25%	**9.07**
300_ieee__sad	MEM		0.8%	5.4	0.12%	7.80	0.13%	**2.22**

Time in seconds

The Determinant Hierarchy

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{bmatrix} \geq 0$$

positive semidefinite = all principal minors are non-negative

principal minor = determinant of the submatrix formed by deleting n-k rows and the corresponding n-k columns, k in {1,…,n}.

The Determinant Hierarchy

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$$k = 1, \; x_{ii} \geqslant 0, \; i \in \{1, 2, 3\}$$

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$$k = 2,\ x_{ii}x_{jj} \geqslant x_{ij}^2,\ (i, j) \in \{(1, 2), (1, 3), (2, 3)\}$$

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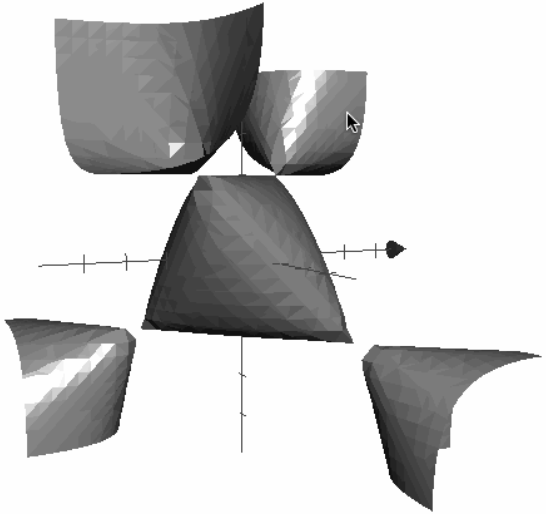
$$k = 3,\ 2x_{12}x_{23}x_{13} + x_{11}x_{22}x_{33} \geqslant x_{12}^2x_{33} + x_{13}^2x_{22} + x_{23}^2x_{11}$$

SDP: Geometric Intuition

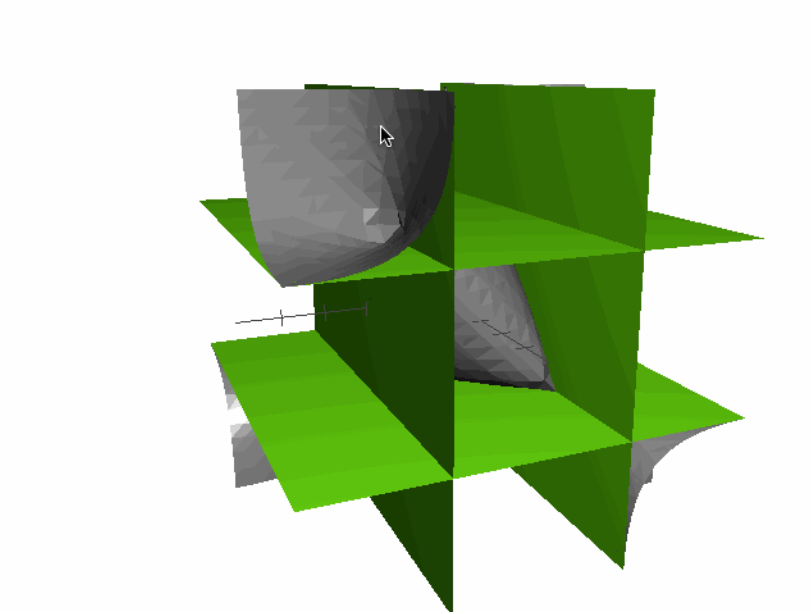


$$2x_{12}x_{23}x_{13} + 1 \geqslant x_{12}^2 + x_{13}^2 + x_{23}^2$$

Feasible region after fixing diagonal elements to one

SDP: Geometric Intuition

Non-convex constraints defining a convex region

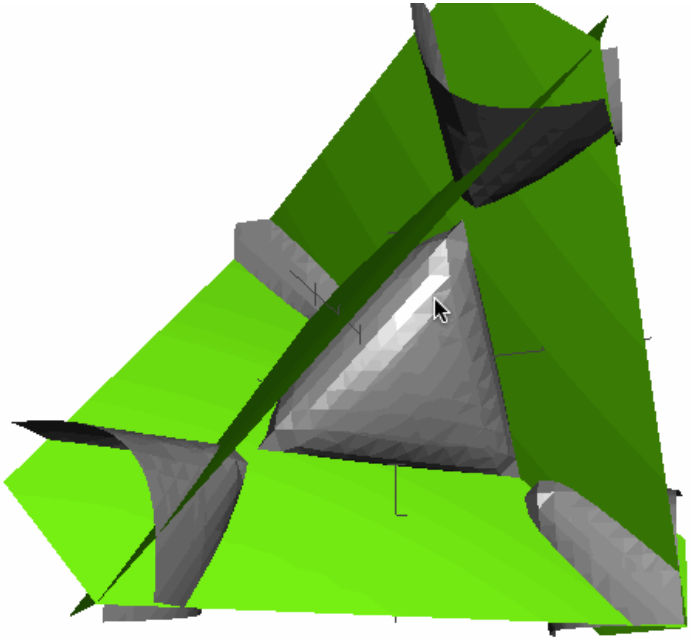


$$2x_{12}x_{23}x_{13} + 1 \geqslant x_{12}^2 + x_{13}^2 + x_{23}^2$$

After adding the 2x2 submatrix determinant constraints

SDP: Geometric Intuition

Can we generate Outer-Approximation Cuts?



Should also work in higher dimensions*

SDP: Geometric Intuition

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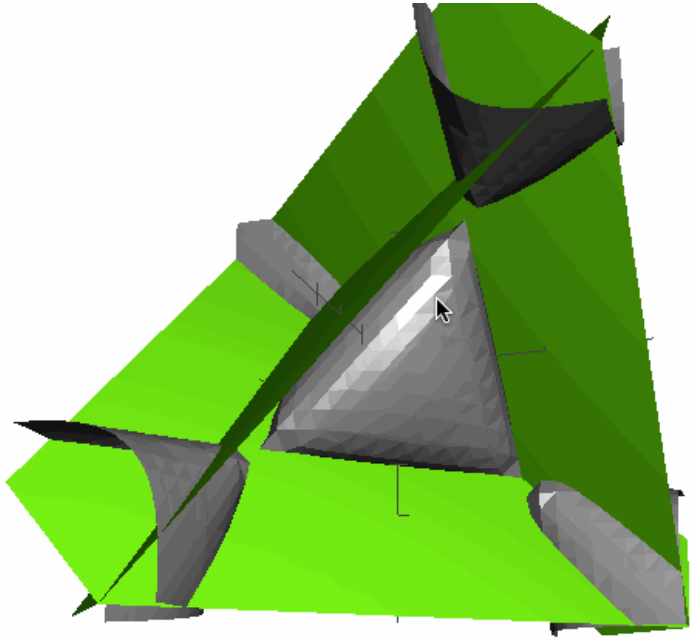


Should also work in higher dimensions*

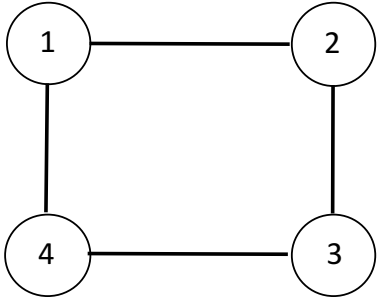
* might need an exponential number of cuts. [Braun et al. 2015].

Chordal Graphs To The Rescue

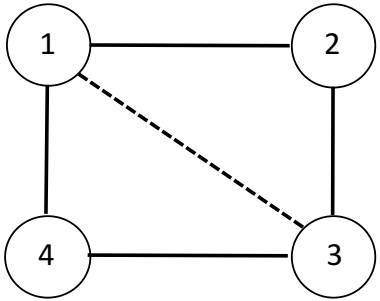
A Graph is chordal if every cycle of length >= 4 has a chord

Chordal Graphs To The Rescue

A Graph is chordal if every cycle of length >= 4 has a chord



Not Chordal



Chordal

Max-Clique Decomposition of Chordal Graphs

Grone et al. 1984

$$X \succeq 0 \equiv X_{\mathcal{C}_i} \succeq 0, \text{ for all max-cliques } \mathcal{C}_i \in G$$

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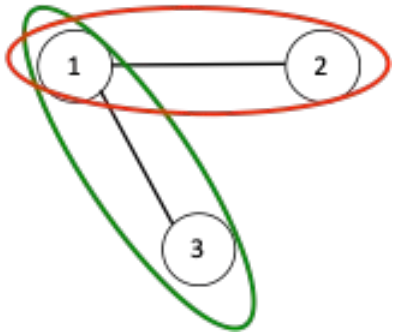

Intuition behind theorem:
No cyclic relationship

$$(x_1 - x_2) + (x_2 - x_3) + (x_3 - x_1) = 0$$

Fast Forward to 2022..

Fast Forward to 2022..

A few months later..

GravitySDP

https://github.com/coin-or/Gravity/tree/GravitySDP

GRAVITY

Mathematical Modeling for Optimization and Machine Learning

GravitySDP

GRAVITY

Uses
Eigen

GravitySDP

GRAVITY

Uses
Eigen

Automatic Sparsity
Detection + Chordal
Graph Completion +
Max Clique Generation +
Auxiliary Variable
Projection

GravitySDP

GRAVITY

Uses
Eigen

Automatic Sparsity
Detection + Chordal
Graph Completion +
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Ties to Cplex and Gurobi
Callback

IBM CPLEX

GravitySDP

GRAVITY

Uses
Eigen

Automatic Sparsity Detection + Chordal Graph Completion + Max Clique Generation + Auxiliary Variable Projection

Ties to Cplex and Gurobi Callback

IBM CPLEX

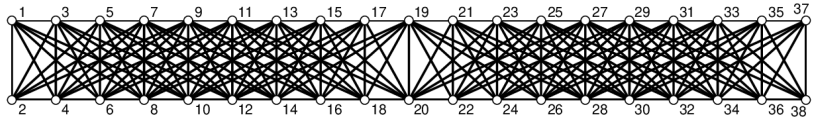
Can run Multi-threading in Cplex!

Numerical Experiments

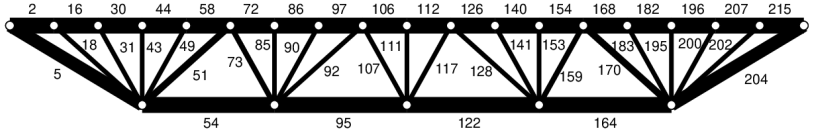
Truss Structure Optimization

SCIP-SDP
4.1.0

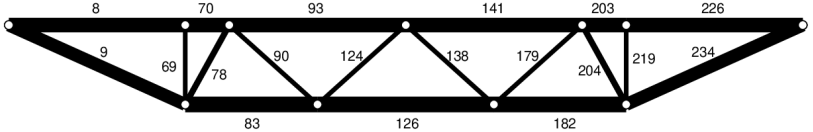
mosek
10



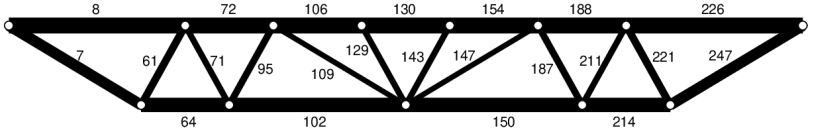
(a) Ground structure with $n_E = 263$, $n_N = 38$, and $n_d = 73$.



(b) Solution with strength constraints, using conventional approach. $W^* = 687.03\,\mathrm{kg}$.



(c) Solution with strength constraints, using the proposed approach. $W^* = 573.20\,\mathrm{kg}$.



(d) Solution with Euler buckling constraints. $W^* = 627.89\,\mathrm{kg}$.

A **truss** is an assembly of *members* such as beams, connected by *nodes*, that creates a rigid structure

	GravitySDP			GravitySDP (no root refine).			SCIPSDP		
Instance	objective	relaxation bound	time (seconds)	objective	relaxation bound	time (seconds)	objective	relaxation bound	time (seconds)
2x5-1scen-12bars	3.51	3.51	350	3.51	3.51	2180	3.51	3.51	233
2x5-2scen-3bars	7.33	7.33	13	7.33	7.33	12	7.33	7.33	48
2x5-2scen-4bars	6.66	6.66	14	6.66	6.66	22	6.66	6.66	83
2x5-3bars	4.79	4.79	32	4.79	4.79	4	4.79	4.79	21
2x6-3bars	6.20	6.20	81	6.20	6.20	27	6.20	6.20	151
2x7-3bars	8.35	8.35	385	8.35	8.35	402	8.35	8.35	1286
3x3-2bars-3scen	33.91	33.91	6	33.91	33.91	11	33.91	33.91	35
3x3-2fixed-8bars	2.56	2.56	23	2.56	2.56	26	2.56	2.56	6
3x3-2scen-6bars	7.86	7.86	28	7.86	7.86	91	7.86	7.86	61
3x3-2scen-8bars	7.74	7.74	28	7.74	7.74	68	7.74	7.74	49
4x5-2bars	6.16	6.16	120	6.16	6.16	22	6.16	6.16	2054
3x3-2scen-small-rob	2.81	2.81	30	2.81	2.81	27	2.81	2.81	52
3x3-3scen-8bars	0.69	0.69	95	0.69	0.69	119	0.69	0.69	102
3x3-5bars-2scen	4.03	4.03	6	4.03	4.03	23	4.03	4.03	8
3x4-1scen-6bars	0.77	0.77	197	0.77	0.77	41	0.77	0.77	90
3x4-1scen-8bars	0.60	0.60	27	0.60	0.60	126	0.60	0.60	22
3x4-2fixed-4bars-nominal	7.18	7.18	24	7.18	7.18	644	7.18	7.18	20
4x3-2bars-3scen	32.21	32.21	166	32.21	32.21	794	32.21	32.21	273
4x4-1bar-2scen	12.16	12.16	1731	12.16	12.16	2759	12.16	10.44	10800
5x5-1bar	8.12	8.12	3420	8.12	8.12	1423	8.12	8.12	7191
bridge-2x5-5bars	2.50	2.50	12	2.50	2.50	29	2.50	2.50	9
2x3-3bars	2.12	2.12	1	2.12	2.12	0	2.12	2.12	1
bridge-2x6-4bars-2scen	6.60	6.60	113	6.60	6.60	207	6.60	6.60	888
bridge-2x7-4bars	9.68	9.68	13	9.68	9.68	87	9.72	9.72	7
bridge-2x8-2bars-2scen	5.31	5.31	954	5.31	5.31	1448	5.31	5.31	814

	GravitySDP			GravitySDP (no root refine).			SCIPSDP		
bridge-2x8-2bars-2scen	5.31	5.31	954	5.31	5.31	1448	5.31	5.31	814
bridge-2x9-2bars-nominal	5.69	5.69	2155	5.69	5.69	4049	5.69	5.69	352
bridge-2x9-2bars	4.66	4.66	5723	4.66	4.61	10801	4.66	4.66	816
bridge-2x10-2bars-2scen	7.09	6.73	10826	7.25	6.38	10802	7.29	6.77	10800
bridge-3x5-4bars-nominal	4.28	4.28	11	4.28	4.28	140	4.28	4.28	2
bridge-3x5-4bars	9.07	9.01	10808	9.07	9.00	10803	-	-	Error
bridge-3x7-2bars-nominal	7.46	7.46	6735	8.66	2.92	10802	7.46	7.46	503
bridge-3x7-2bars	10.15	10.15	548	10.07	3.49	10803	10.15	10.15	49
2x4-2scen-3bars	5.33	5.33	8	5.33	5.33	6	5.33	5.33	36
bridge-3x8-1bar-2scen	-	12.88	10861	-	5.04	10801	18.45	18.45	178
bridge-3x9-2bars	-	6.83	10895	-	2.78	10803	14.50	14.49	10800
demonst-1bar-3scen	22.81	22.81	1081	22.81	22.81	486	22.81	21.33	10800
demonst-2bars-2scen	10.13	10.13	95	10.13	10.13	97	10.13	10.13	4481
demonstsmall-1bar-4scen	18.49	18.49	5	18.49	18.49	7	18.49	18.49	51
demonstsmall-2bar-3scen	3.58	3.58	3	3.58	3.58	2	3.58	3.58	27
demonstsmall-2bars-2scen	7.30	7.30	3	7.30	7.30	7	7.30	7.30	41
demonstsmall-5bar-1scen-nominal	0.97	0.97	2	0.97	0.97	0	0.97	0.97	1
test-bridge2	6.89	6.89	16	6.89	6.89	67	6.89	6.89	108
test-bridge3	4.59	4.59	15	4.59	4.59	13	4.59	4.59	25
2x4-2scen-6bars	3.97	3.97	10	3.97	3.97	16	3.97	3.97	15
2x4-3bars-nominal	3.83	3.83	3	3.83	3.83	1	3.83	3.83	8
2x4-3bars	3.08	3.08	2	3.08	3.08	1	3.08	3.08	3
2x4-8bars-2scen	2.03	2.03	25	2.03	2.03	15	2.03	2.03	94
2x4-16bars	0.62	0.62	4	0.62	0.62	17	0.62	0.62	13
2x5-1scen-6bars	3.73	3.73	132	3.73	3.73	754	3.73	3.73	166

Geometric Mean	
Method	Time (secs)
GravitySDP	75
GravitySDP (no root refine)	103
SCIPSDP	133

Geometric Mean	
Method	Time (secs)
GravitySDP	75
GravitySDP (no root refine)	103
SCIPSDP	133
Best	35

Numerical Experiments

Algebraic Connectivity

Algebraic connectivity can measure how weakly any subset of vertices is connected to the remaining graph:

- Minimize latency in communicating data/information across the network
- Bottlenecks can only occur at higher data rates and robustness to node and link failures

LaplacianOpt

A Julia Package for Maximizing Algebraic Connectivity of Graphs

Status: CI passing | codecov 93% | Documentation passing

Numerical Experiments

10x improvement in runtime, closed 6 open instances.

Main advantages:

- projected SDP cuts
- Multi-threaded cut generation

LaplacianOpt

A Julia Package for Maximizing Algebraic Connectivity of Graphs

Status: CI passing | codecov 93% | Documentation passing

Thank you!

https://lanl-ansi.github.io/